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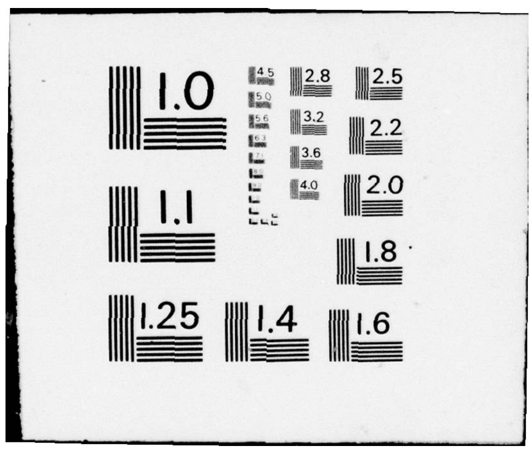
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PARALLEL AND INTERSECTING FLATS FRACTIONS: ESTIMABILITY AND ALIAS STRUCTURES

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ABSTRACT

Regular fractional factorial designs for the s^n, $s = p^\alpha$, factorial may be obtained as solutions sets to linear equations of the form $A\underline{t} = \underline{c}$ over GF(s). It is well known how to determine which factorial effects are estimable, and how to construct the alias sets directly from the system matrix, A. Consider the k sets of equations over GF(s).

$$A_i\underline{t} = \underline{c}_i, \quad i=1,2,\ldots,k$$

where A_i is $m_i \times n$ of rank m_i, and let $T = \bigcup_{i=1}^{k} \{\underline{t} | A_i\underline{t} = \underline{c}_i\}$. If $A_1 = A_2 = \ldots = A_k$, T is called a parallel flats fraction, and if not all A_i are equal, T is called an intersecting flats fraction. There is no unified theory developed for determining estimability and alias structures from ~~these~~ types of fractions involving intersecting flats. The purpose of this paper is to present some preliminary results in the development of such a theory.

1. Fractional Replication

As the number of factors in an s^n experiment increases, the high cost of experimentation, expecially in industry, often prohibits use of a complete factorial design. If economy of time, space, and materials is an important consideration, and if it is reasonable to assume that certain higher order interactions are negligible, then a fractional factorial design may be appropriate. The concept of fractional replication for 2^n and 3^n experiments was introduced by Finney (1945) as a logical outgrowth of the theory of confounding, and since that time the theory of fractional replication has developed rapidly. Experimenters involved in high cost research have found the designs to be useful, particularly as screening experiments run on a large number of factors.

The notion of linear flats within a finite Euclidean geometry, as developed by Bose and Kishen (1940) and Bose (1947), is a useful device for partitioning degrees of freedom and constructing fractional factorial designs. Suppose $s = p^\alpha$, where p is prime, and consider an s^n factorial experiment involving n factors, each at s levels. Let every possible treatment combination or "run" be represented as an n x 1 vector $\underline{t}$ with elements from a Galois field of order s, denoted GF(s), and with ith element corresponding to the level of the ith factor. Then there clearly exists a one-to-one correspondence between the s^n possible treatment combinations and the s^n points in the Euclidean geometry of order n over the field of order s, denoted as

$$EG(n, s) = \{(t_1, t_2, \ldots, t_n) \mid t_i \in GF(s), i = 1, \ldots, n\}.$$

A linear $(n - k)$-flat in $EG(n, s)$ is defined to be the set of points in $EG(n, s)$ that satisfy a consistent system of k independent linear equations, each of the form $\sum_{i=1}^{n} a_i t_i = a_r$, where $a_0 = 0, a_1, \ldots, a_{s-1}$ denote the elements of $GF(s)$ and $a_r = a_i$ for some $i = 0, 1, \ldots,$ or $s - 1$.

Consider a pencil of s parallel $(n - 1)$-flats with equations of the form $\sum_{i=1}^{n} a_i t_i = a_r$, where a_i is fixed, $i = 1, \ldots, n$, and a_r assumes consecutively the s values $a_0, a_1, \ldots, a_{s-1}$. Any such pencil partitions the s^n points in $EG(n, s)$ into s subsets of s^{n-1} points each, with contrasts between these subsets accounting for $s - 1$ degrees of freedom. Thus the pencil $t_i = a_r$, i fixed, and $r = 0, 1, \ldots, s - 1$, is used to define the $(s - 1)$ degrees of freedom corresponding to the main effects of the ith factor, the pencils given by equations of the form $t_i + a_u t_{i'} = a_r$, i and i' fixed, $u = 1, 2, \ldots, s - 1$ and $r = 0, 1, \ldots, s - 1$, are used to define the $(s - 1)^2$ degrees of freedom corresponding to the two-way interaction of Factors i and i', and so on. Suppose, for instance, that each factor in a 3^n experiment appears at levels 0, 1, and 2. The two degrees of freedom provided for the main effects of the ith factor by the pencil $t_i = r$, $r = 0, 1, 2$, correspond to the linear contrast, written as $\{\underline{t} \mid t_i = 2\} - \{\underline{t} \mid t_i = 0\}$, and to the quadratic contrast, written as $\{\underline{t} \mid t_i = 2\} - 2\{\underline{t} \mid t_i = 1\} + \{\underline{t} \mid t_i = 0\}$. For any two factors F_i and $F_{i'}$, the same type of contrasts may be used to partition the two degrees of freedom for interaction $F_i F_{i'}$ specified by the pencil $t_i + t_{i'} = r$, $r = 0, 1, 2$, and the two degrees for $F_i F_{i'}^2$, specified by the pencil $t_i + 2t_{i'} = r$, $r = 0, 1, 2$. This procedure gives a single degree of freedom, breakdown of the four degrees of freedom corresponding to the interaction of Factors i and i'. Clearly, the

components of this breakdown are not the same as the components of the breakdown into single degrees of freedom for linear x linear, linear x quadratic, and so on, that was described in Section 1.1.

The simplest kind of fractional factorial design for the s^n experiment consists of all treatment combinations corresponding to points in EG(n, s) that lie on a particular (n - 1)-flat in a pencil of s parallel flats. The aliasing structure for such a design is easily determined by taking linear combinations of the defining relationship for the selected pencil, say $\sum_{i=1}^{n} a_i t_i = a_r$, $r = 0, 1, \ldots, s - 1$, with the defining relationships for main effects ($t_i = a_r$, $r = 0, 1, \ldots, s - 1$), two-factor interactions ($t_i + a_u t_{i'} = a_r$, $r = 0, 1, \ldots, s - 1$), and so on. In general, the defining pencil should be selected so that effects of interest are aliased, or confounded, with negligible effects.

For example, one fraction of a 3^4 experiment with Factors F_1, F_2, F_3, and F_4, each at levels 0, 1, and 2, consists of the 27 points on the three-flat specified by $t_1 + 2t_2 + t_3 + 2t_4 = 0$. Equivalently, this fraction is often represented by the defining relations

$$I = F_1F_2^2F_3F_4^2 \; (= F_1^2F_2F_3^2F_4).$$

In this design the $F_2F_3^2$ interaction, which by definition is based on contrasts among sums of points for which $t_2 + 2t_3$ = constant, is aliased with $F_1F_4^2$ since $(t_1 + 2t_2 + t_3 + 2t_4) + (t_2 + 2t_3) = t_1 + 2t_4$ = constant. Similarly, the F_1F_3 interaction is aliased with the F_2F_4 interaction since $(t_1 + 2t_2 + t_3 + 2t_4) + 2(t_1 + t_3) = 2(t_2 + t_4)$. If three-factor and higher order interactions are negligible, the alias sets for this

fraction are $\{\mu\}$, $\{F_1\}$, $\{F_2\}$, $\{F_3\}$, $\{F_4\}$, $\{F_1F_2\}$, $\{F_1F_4\}$, $\{F_3F_4\}$, $\{F_2F_3\}$, $\{F_1F_3,\ F_2F_4\}$, $\{F_1F_3^2\}$, $\{F_2F_4^2\}$, $\{F_1F_2^2,\ F_3F_4^2\}$, $\{F_1F_4^2,\ F_2F_3^2\}$.

In general, regular s^{n-k} fractions of the s^n factorial experiment consist of solutions over GF(s) to

$$A\underline{t} = \underline{c} \tag{1}$$

where A is a k x n matrix of rank k and $\underline{c}$ is k x 1, both over a Galois field of order s. Geometrically, the equations (1) represent a linear (n - k)-flat of s^{n-k} points in EG(n, s). If $\sum_{i=1}^{n} a_i t_i = a_r$ and $\sum_{i=1}^{n} a_i' t_i = a_r'$ are two equations used to define a fraction, their generalized interaction is specified by $(\sum_{i=1}^{n} a_i t_i) + a_u(\sum_{i=1}^{n} a_i' t_i) = a_w$, $u = 1, \ldots, s-1$, $w = 0, \ldots, s-1$, and it is an easy matter to determine the alias sets for the fraction by considering all linear combinations involving defining relationships and their generalized interactions.

In practice a regular s^{n-k} fraction is often specified by defining relations in k "words", chosen so that effects of interest are aliased with higher order, negligible interaction effects. For example, a 3^{6-2} fraction consisting of the $N = 3^4 = 81$ treatment combinations $\underline{t}$ that are solutions to

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{bmatrix} \underline{t} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \pmod 3$$

can be specified by the defining relations

$$I = F_1F_2F_3F_4 = F_3F_4^2F_5F_6 = F_1F_2F_3^2F_5F_6,$$

where $F_1F_2F_3^2F_5F_6$ is the generalized interaction generated by $F_1F_2F_3F_4$ and $F_3F_4^2F_5F_6$. The resulting fraction is a design in which main effects are aliased with three-factor and higher order interactions. Thus, if three-factor and higher order interactions are negligible, the design provides estimates of main effects in the presence of non-negligible two-factor interactions.

Bose (1944) defined a linear combination $\underline{\lambda}'\underline{\beta}$ of the parameters in a linear model to be estimable if and only if there exists a linear combination $\underline{c}'\underline{Y}$ of the observations such that $E(\underline{c}'\underline{Y}) = \underline{\lambda}'\underline{\beta}$. A complete factorial design provides an estimate for every single degree of freedom component of $\underline{\beta}$. With a fractional factorial design, on the other hand, $\beta_i \in \underline{\beta}$ is estimable if and only if the column of X that is multiplied by β_i in the model $E(\underline{Y}) = X\underline{\beta}$ is linearly independent of all remaining columns of X.

Box and Hunter (1961), in a paper dealing with regular 2^{n-k} fractions of the 2^n factorial, introduced the term resolution of a design as a means of classifying fractional factorial designs on the basis of estimability of effects. The term applies directly to the regular s^{n-k} fractions of the s^n factorial experiment, in which case the design is resolution r if all linear combinations of the rows of A from equation (1) have at least r nonzero coordinates. The names resolution III, resolution IV, resolution V, and so on, make obvious reference to the particular kind of fractions considered by Box and Hunter. For example, a resolution IV design for the s^n factorial experiment requires each

word in the defining relations to have at least four letters and at least one word to have exactly four letters. Such a design permits estimation of the general mean and all main effects if all three-factor and higher order interactions are negligible. Further, the estimates are uncorrelated with each other.

To the extent that the name resolution r still carries implication to the regular s^{n-k} fractions of the s^n experiment it is an unfortunate terminology. A slightly modified definition which is now generally accepted was given by Webb (1965, 1968) and Margolin (1969a) as follows.

Definition 1. If a design is such that all effects (main effects, interactions) involving r or fewer factors are estimable, ignoring all interactions of r + 1 or more factors, the design is said to be of resolution 2r + 1; if all effects involving r - 1 or fewer factors are estimable, ignoring all interactions of r + 1 or more factors, the design is said to be of resolution 2r.

Note that this definition refers only to the estimability of effects, not to any specific method of construction. Further, it does not require estimates to be uncorrelated with each other or μ to be estimable. In general, designs of odd resolution permit estimation of all effects not assumed to be zero, while designs of even resolution permit estimation of certain effects in the presence of other non-zero, nonestimable effects. In practice, designs of resolutions III, IV, and V are of the most interest. A resolution III design allows estimation of main effects when two-factor and higher order interactions are negligible, and a resolution V design allows estimation of main effects and two-factor interactions when three-factor and higher order interactions are

negligible. A resolution IV design, on the other hand, permits estimation of main effects in the presence of nonestimable two-factor interactions when three-factor and higher order interactions are negligible.

This paper was motivated by the authors' search for resolution IV designs for the S^n factorial (Anderson and Thomas 1975a, b). This search led to the consideration of parallel and intersecting flats construction. At the present time there is no unified theory developed for determining the alias sets, or even which effects are estimable, from such constructions. The purpose of this paper is to begin a development of estimability and alias structure theory for parallel and intersecting flat fractions. Because of the initial motivation most of the examples relate to estimability of main effects in the presence of two-factor interactions.

2. Construction Using Intersecting Flats

In Section 1 a regular s^{n-k} fraction of the s^n factorial experiment was defined to be the set of solutions over GF(s) to

$$A\underline{t} = \underline{c}, \tag{3}$$

where A is a k x n matrix of rank k and $\underline{c}$ is k x 1, both over a field of

order s. Thus there is a one-to-one correspondence between the s^{n-k} points of EG(n,s) that lie on the linear (n - k)-flat specified by (2) and the s^{n-k} treatment combinations that are in the fraction determined by (2). An alternative to choosing points on a single flat in EG(n,s) is to take the union of the points on several flats. Thus consider the flats generated by equations

$$A_i\underline{t} = \underline{c}_i, \quad i = 1,2,\ldots,r \tag{3}$$

where A_i is $m_i \times n$ of rank m_i and $\underline{c}_i$ is $m_i \times 1$. The design T corresponding to (3) is

$$T = \bigcup_{i=1}^{r} \{\underline{t} \mid A_i\underline{t} = \underline{c}_i\} \quad . \tag{4}$$

The i^{th} flat contains s^{n-m_i} points, but since the flats may intersect in various ways the number of points in T as well as the estimability of factorial effects depend on the A_i and $\underline{c}_i$ in a rather complex manner.

Consideration of designs of type (4) was motivated by a search for a general series of minimal or near minimal resolution IV designs for the s^n factorial. Theorems 1 and 2 produce such a series in $N = s(s - 1)n$ runs, only $s(s - 2)$ more than the perhaps unattainable lower bound.

Theorem 1. Let $A_1, A_2, \ldots, A_n$ be $(n - 2) \times n$ matrices of rank $(n-2)$ such that for $i = 1, 2, \ldots, n$,

1. the i^{th} column of A_i is $\underline{0}$,
2. one column of A_i has all nonzero elements, and
3. the remaining $(n - 2)$ columns of A_i are some permutation of $I_{(n-2)}$.

Then the fraction

$$T = \bigcup_{i=1}^{n} \{\underline{t} \mid A_i\underline{t} = \underline{c}_i\} \tag{5}$$

is resolution IV for any choice of the $\underline{c}_i$.

Proof. Each row of A_i contains exactly two nonzero elements and has ith element zero. Further, any linear combination of two or more rows of A_i will have at least two nonzero elements and a zero in the ith position. It follows directly that in the ith set of equations, the main effect of the ith factor is aliased only with three factor or higher order interactions. Thus the main effects of the ith factor are estimable from the runs corresponding to

$$A_i\underline{t} = \underline{c}_i, \; i = 1, 2, \ldots, n,$$

so all main effects are estimable from the union

$$T = \bigcup_{i=1}^{n} \{\underline{t} \mid A_i\underline{t} = \underline{c}_i\}.$$

The proof is complete.

Example 1. The design T formed by taking the union of solutions over GF(s) to the following five systems of equations is an s^5 resolution IV design in s(s - 1)5 runs.

Flat 1.

$$\begin{bmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \underline{t} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Flat 2.

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \underline{t} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Flat 3.

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \underline{t} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Flat 4.

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \underline{t} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Flat 5.

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{bmatrix} \underline{t} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The union of the points on these five flats with s = 3 is

$$T = \begin{bmatrix} 012 & 012 & 012 & 012 & 012 & 012 & 012 & 012 & 012 & 012 \\ 012 & 201 & 120 & 201 & 012 & 012 & 012 & 012 & 012 & 012 \\ 012 & 201 & 120 & 120 & 120 & 201 & 012 & 012 & 012 & 012 \\ 012 & 201 & 120 & 120 & 120 & 120 & 120 & 201 & 012 & 012 \\ 012 & 201 & 120 & 120 & 120 & 120 & 120 & 120 & 120 & 201 \end{bmatrix}$$

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-----1-----
          -----2-----
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where the underscoring indicates the treatment combinations generated by the ith system of equations, i = 1, 2, 3, 4, 5.

Theorem 2. For $s = p^{\alpha}$, p prime, there exist fractions of resolution IV in $N = s(s - 1)n$ runs for the s^n factorial.

Proof. The proof is by construction. Let the column of all nonzero elements of the A_i in Theorem 1 be $-\underline{1}$, that is, an (n - 2) column with every element $-1 \in GF(s)$. Further, let the column $-\underline{1}$ be the second column in A_1 and the first in $A_2, A_3, \ldots, A_n$. Finally, arrange the rows of each A_i so that the set of (n - 2) columns, each with a single one and (n - 3) zeros, in order form I_{n-2}. The arrangement in Example 3.1 illustrates this particular ordering. The $\underline{c}_i$, i = 1, ..., n, are then selected so that flats 1 and n intersect in s points, flats i and (i + 1), i = 1, ..., n - 1, intersect in s points, and all remaining intersections of pairs of flats are empty. The construction is represented pictorially in Figure 1.

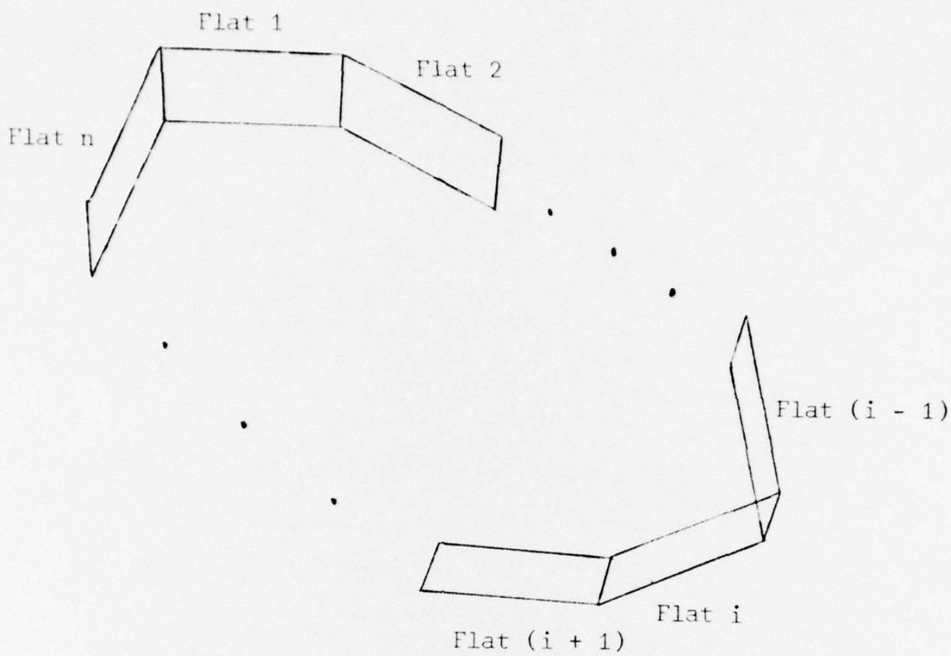

Figure 1. Geometric Representation of Construction for Design in $s(s - 1)n$ Runs

With $\underline{c}_i = (c_{1i}, c_{2i}, \ldots, c_{n-2,i})'$, $i = 1, \ldots, n$, the construction is achieved as follows.

1. $\underline{c}_2 - \underline{c}_1 = a\underline{1}$ for some constant a implies flats 1 and 2 intersect in s points.

2. $c_{k+1,n} = c_{k1} + c_{1n}$, $k = 1, \ldots, n - 3$ implies flats 1 and n intersect in s points.

3. $c_{ki} = c_{k,i+1}$, $k = 1, \ldots, i - 2, i, \ldots, n - 2$, and $i = 3, \ldots, n - 1$, implies flats i and $(i + 1)$ intersect in s points, $i = 1, \ldots, n - 1$.

4. $c_{n-2,i} \neq c_{n-2,1} + c_{1i}$, $i = 3, \ldots, n - 1$ implies flat 1 does not intersect with flat i, $i = 3, \ldots, n - 1$.

5. $c_{ii'} = c_{i-1,i}$, $i = 2, \ldots, n-2$, and $i' = i + b$, $b > 1$, implies flat i does not intersect with flat $i + b$, $b > 1$, $i = 2, \ldots, n-2$.

For arbitrary $\underline{c}_1 = [c_{11}, \ldots, c_{n-2,1}]'$ and c_{12}, and any $c_{1n} \neq c_{12} - c_{11}$, the following construction clearly satisfies conditions 1 - 3.

$$c_{k2} = c_{12} + c_{k1} - c_{11}, \; k = 3, \ldots, n-2$$

$$c_{1i} = c_{1n}, \; i = 3, \ldots, n-1$$

$$c_{ki} = c_{k,i-1}, \; k = i-1, \ldots, n-2, \; c_{1n} + c_{k-1,1}, \; k = 2, \ldots, i-2, \; i = 3, \ldots, n.$$

Also, since $c_{1n} \neq c_{12} - c_{11}$ implies $c_{1n} \neq c_{n-2,2} - c_{n-2,1}$

implies $c_{1i} \neq c_{n-2,i} - c_{n-2,1}$, $i=3, \ldots, n-1$,

implies $c_{n-2,i} \neq c_{n-2,1} + c_{1i}$, $i = 3, \ldots, n-1$,

condition 4 is satisfied. And for planes i and $i' = i + b$, $b > 1$, $i = 2, \ldots, n-2$, $c_{1n} \neq c_{12} - c_{11}$ implies $c_{i-1,1} + c_{1n} \neq c_{i-1,1} + (c_{12} - c_{11})$

implies $c_{1n} + c_{i-1,1} \neq c_{i-1,2}$

implies $c_{ii'} \neq c_{i-1,i}$,

so condition 5 is satisfied.

One such choice of the $\underline{c}_i$ is $\underline{c}_1 = \underline{c}_n = \underline{0}$, $\underline{c}_2 = \underline{1}$, $\underline{c}_3 = (0, 1, 1, \ldots, 1)'$, $\underline{c}_4 = (0, 0, 1, \ldots, 1)'$, $\ldots$, $\underline{c}_{n-1} = (0, 0, 0, \ldots, 1)'$.

Example 2. The 3^5 design in Example 1 illustrates the construction of Theorem 2.

The designs constructed by Theorem 2 have a convenient representation in terms of parallel one-flats. Since flats 1 and n, as well as flats i and i+1, i=1,2,...,n-1, intersect in s points, each of the

system of equations

$$\begin{bmatrix} A_1 \\ A_n \end{bmatrix} \underline{t} = \begin{bmatrix} \underline{c}_1 \\ \underline{c}_n \end{bmatrix}, \text{ and } \begin{bmatrix} A_i \\ A_{i+1} \end{bmatrix} \underline{t} = \begin{bmatrix} \underline{c}_i \\ \underline{c}_{i+1} \end{bmatrix}, \quad i = 1, 2, \ldots, n-1, \tag{6}$$

is of rank n - 1 and consistent. In fact, each can be reduced to the form

$$\begin{bmatrix} -1 & & \\ & \cdot & \\ -1 & & \\ \vdots & \cdot & I_{n-1} \\ \vdots & & \\ & \cdot & \\ -1 & & \end{bmatrix} \underline{t} = \underline{d} \ . \tag{7}$$

The equation (7) with the $(i - 1)^{th}$ row eliminated, $i = 2, 3, \ldots, n$, are precisely

$$A_i \underline{t} = \underline{d}^* \ ,$$

where $\underline{d}^*$ is the $(n - 2) \times 1$ vector obtained from $\underline{d}$ by eliminating the $(i - 1)^{th}$ element. In relation to Theorem 2 $\underline{d}^* = \underline{c}_i$. Thus the set of points on the i^{th}, $i = 2, \ldots, n$, flat may be represented as s parallel one-flats defined by

$$\begin{bmatrix} -1 & \cdot & \\ -1 & & \\ \vdots & \cdot & I_{n-1} \\ \vdots & \cdot & \\ -1 & & \end{bmatrix} \underline{t} = \underline{d}_k, \quad k = 0, 1, 2, \ldots, s-1, \tag{8}$$

where the $(i - 1)^{th}$ element of $\underline{d}_k$ is $x_k \in GF(s)$ and the remaining n - 2 elements are $\underline{d}^* = \underline{c}_i$. Similarly, if the first equation of (7) is subtracted from each of the remaining, the result is

$$A_1\underline{t} = \begin{bmatrix} d_2 & - d_1 \\ d_3 & - d_1 \\ & \vdots \\ d_{n-1} & - d_1 \end{bmatrix} = \underline{c}_1 , \tag{9}$$

and the s^2 points on $A_1\underline{t} = \underline{c}_1$ are obtained by keeping $\underline{c}_1$ fixed and letting d_1 take all values in GF(s). Thus in the parallel flats representation (7) the coefficient matrix is fixed while vector $\underline{d}$ is varied.

Example 3. The parallel flats representation of the 3^5 design of Example 1 is given by

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \underline{t} = \underline{d}_i, \quad i = 1, 2, \ldots, 10$$

where the $\underline{d}_i$ are the columns of

$$D = \begin{bmatrix} 0 & 2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

The one-flats of each plane are identified by the spacing in the array T of Example 1. Several additional examples are included in Section 3.

3. The 3^4 Experiment - More Examples.

A somewhat detailed consideration of the 3^4 experiment will illustrate both the versatility and the complexity of design construction from intersecting linear flats.

For the 3^4 factorial consider first

$$A_1 = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{bmatrix}, \tag{10}$$

as specified in Theorems 2 and 3 of Section 2,

$$\underline{c}_i = (c_{1i}, c_{2i})', \quad i = 1, 2, 3, 4,$$

and

$$T = \bigcup_{i=1}^{4} \{\underline{t} \mid A_i \underline{t} = \underline{c}_i\} .$$

The number of runs in T, which corresponds to the number of points in the geometric representations of T, clearly depends on the choice of $\underline{c}_i$, i = 1, 2, 3, 4. For instance, if $\underline{c}_i = \underline{0}$, i = 1, 2, 3, 4, the design includes the 27 runs represented in Figure 2 of Section 3. On the other hand, if the $\underline{c}_i$ are chosen to be

$\underline{c}_1' = (0,0)$

$\underline{c}_2' = (1,1)$

$\underline{c}_3' = (0,1)$

$\underline{c}_4' = (0,0)$

the result is a design in 24 runs. In general, with A_1, A_2, A_3, and A_4 defined as above, any choice of $\underline{c}_i$ yields a two-flat which is simply the union of three non-intersecting one-flats, each being of the form $\{\underline{t}, \underline{t} + \underline{1}, \underline{t} + (2)\underline{1}\}$. Thus a design $T = \bigcup_{i=1}^{4} \{\underline{t} \mid A_i \underline{t} = \underline{c}_i\}$, expressed here as the union of four intersecting or nonintersecting two-flats, could alternatively be written as a union of parallel one-flats. That is,

$$T = \bigcup_{i=1}^{4} \{\underline{t} \mid A_i \underline{t} = \underline{c}_i\} = \bigcup_{i=1}^{r} \{\underline{t} \mid B\underline{t} = \underline{d}_i\} ,$$

where B is a 3 x 4 matrix of rank 3 and r depends on the intersection structure of the two-flats. As in Section 2, it is convenient to let

$$B = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

The examples which follow further illustrate the dependence of the number of points in T on the choice of $\underline{c}_1$, $\underline{c}_2$, $\underline{c}_3$, and $\underline{c}_4$. For each example it is assumed that A_i, i = 1, 2, 3, 4, and B are as defined in (10 and (11), respectively. Thus $C = [\underline{c}_1\ \underline{c}_2\ \underline{c}_3\ \underline{c}_4]$ completely specifies a design as a union of two-flats, while $D = [\underline{d}_1\ \underline{d}_2\ \ldots\ \underline{d}_r]$ specifies it as a union of one-flats. In the geometric configurations lines represent two-flats and points represent one-flats.

<u>Example 3.</u> 36 run design $C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 2 & 0 & 2 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 012012111000 \\ 012000012111 \\ 201000222012 \end{bmatrix}$

<u>Figure 2</u>. Geometric Representation of 36 Run 3^4 Design

<u>Example 4.</u> 33 run design $C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 01212111000 \\ 01200012111 \\ 01200222012 \end{bmatrix}$

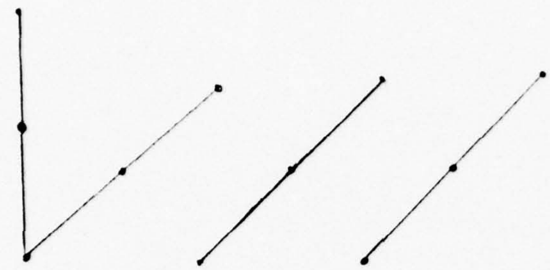

Figure 3. Geometric Representation of 33 Run 3^4 Design

Example 5. 30 run design $C = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ $D = \begin{bmatrix} 0121211000 \\ 0120012222 \\ 0120000012 \end{bmatrix}$

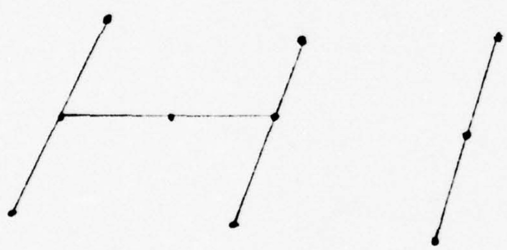

Figure 4. Geometric Representation of 30 Run 3^4 Design

Example 6. 27 run design $C = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ $D = \begin{bmatrix} 012121111 \\ 012001222 \\ 012000012 \end{bmatrix}$

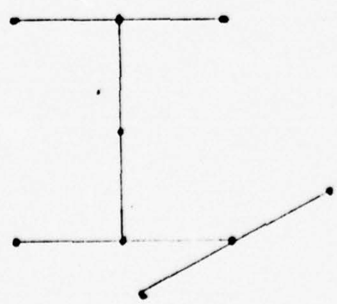

Figure 5. Geometric Representation of 27 Run 3^4 Design

Clearly, this 27 run design is not the same as the 27 run design pictured in Figure 2 of Section 3.1.

There is no uniqueness to the particular constructions presented in Theorems one and two of Section 2. Other choices of the A_i may also give rise to resolution IV designs. For example, the 27 run 3^4 foldover designs, [given by Anderson and Thomas (1975a)], all have representations as four intersecting two-flats. The A_i, i=1, 2, 3, 4, for these designs are listed in Example 7. In each case $\underline{c}_1 = \underline{c}_2 = \underline{c}_3 = \underline{c}_4 = \underline{0}$.

Example 7. A_i matrices for 27 run 3^4 resolution IV Foldover Designs

(a) Design 2 (orthogonal array of strength 3)

$$A_1 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

(b) Design 3

$$A_1 = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

(c) Design 4

$$A_1 = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

The smallest known designs for the 3^4 experiment are the ones generated by Theorem two of Section 2. Since these designs require N = 24 runs while Margolin's lower bound for resolution IV 3^4 designs is $N \geq 21$,

it is of interest to investigate the possibility of using four intersecting two-flats to construct a 21 run resolution IV design.

Consider the 21-point configuration pictured in Figure 6.

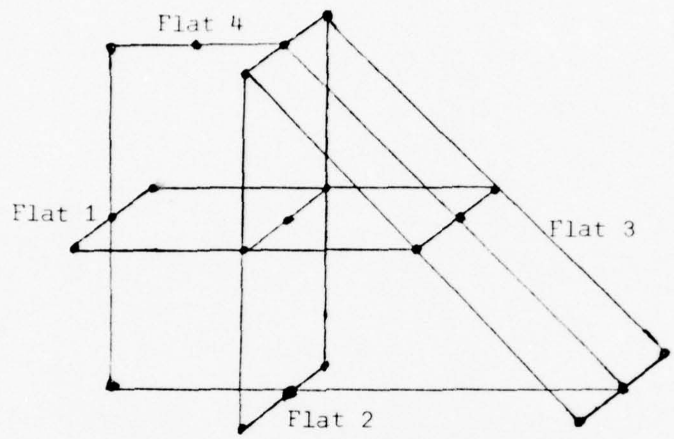

Figure 6. 21-point Configuration in EG(4, 3).

In Figure 6 every pair of two-flats intersect in exactly three points, and every triple of two-flats, except the one consisting of flats 1, 2, and 3, intersect in exactly one point. The intersection of flats 1, 2, and 3 is empty. For i = 1, 2, 3, and 4 the ith flat consists of solutions over GF(3) to

$$A_i \underline{t} = \underline{c}_i \tag{12}$$

where A_i is 2 x 4 of rank two and $\underline{c}_i$ is 2 x 1. Thus any two of the four flats, say flats i and j, intersect in exactly three points if

$$\text{rank} \begin{bmatrix} A_i \\ A_j \end{bmatrix} = \text{rank} \left[\begin{array}{c|c} A_i & \underline{c}_i \\ A_j & \underline{c}_j \end{array}\right] = 3, \tag{13}$$

and any three of the four flats, say flats i. j, and k, intersect in

exactly one point if

$$\text{rank}\begin{bmatrix} A_i \\ A_j \\ A_k \end{bmatrix} = \text{rank}\left[\begin{array}{c|c} A_i & \underline{c}_i \\ A_j & \underline{c}_j \\ A_k & \underline{c}_k \end{array}\right] = 4. \qquad (14)$$

Flats 1, 2, and 3 have an empty intersection if and only if

$$\text{rank}\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = 3 \text{ and rank}\left[\begin{array}{c|c} A_1 & \underline{c}_1 \\ A_2 & \underline{c}_2 \\ A_3 & \underline{c}_3 \end{array}\right] = 4. \qquad (15)$$

In accordance with Theorem 1 of Section 2 let A_i, i = 1, 2, 3, 4, be 2 x 4 matrices of rank 2 such that the ith column of A_i is $\underline{0}$, one column of A_i, denoted by $\underline{a}_i = (a_{1i}, a_{2i})'$, has all nonzero elements, and the remaining two columns of A_i form I_2. Since Theorem 1 guarantees that a fraction $T = \bigcup_{i=1}^{4} \{\underline{t} \mid A_i\underline{t} = \underline{c}_i\}$ constructed using these A_i as coefficient matrices is resolution IV, it is of primary interest to know whether for i = 1, 2, 3, 4 there is some choice of $\underline{a}_i = (a_{1i}, a_{2i})'$ and $\underline{c}_i = (c_{1i}, c_{2i})'$ from GF(3) that will yield the construction of Figure 6. The result, of course, would be a minimal resolution IV design for the 3^4 experiment. Unfortunately, if the A_is are defined as in Theorem 1 if condition (13) is satisfied for every pair of two-flats, and if condition (14) is satisfied for flats i, j, and k when (i, j, k) = (1, 2, 4), (1, 3, 4), and (2, 3, 4), then it is impossible for flats 1, 2, and 3 to satisfy condition (15).

Since the A_is specified by Theorem 1 cannot be used to produce the configuration of Figure 6, consider instead

$$A_i = \begin{bmatrix} 1 & 0 & a_{1i} & b_{1i} \\ 0 & 1 & a_{2i} & b_{2i} \end{bmatrix} \qquad (16)$$

and the corresponding two-flats in EG(4, 3) defined by $A_i\underline{t} = \underline{c}_i$, $i = 1, 2, 3, 4$. With the A_is defined in this way it is possible to select from GF(3) values for $\underline{a}_i = (a_{1i}, a_{2i})'$, $\underline{b}_i = (b_{1i}, b_{2i})'$, and $\underline{c}_i = (c_{1i}, c_{2i})'$, $i = 1, 2, 3, 4$, so that the resulting two-flats intersect as in Figure 6. However, an exhaustive consideration of all possibilities yielded only two intrinsically different constructions of this kind, the conditions for which are discussed below.

The first way to produce Figure 6 is to select $\underline{a}_i$, $\underline{b}_i$, and $\underline{c}_i$, $i = 1, 2, 3, 4$, which satisfy the following conditions.

$$\begin{array}{ll}
(1) & a_{11} \neq a_{12},\ a_{13} = a_{14},\ a_{11} \neq a_{13},\ a_{12} \neq a_{13} \\
(2) & a_{21} = a_{22} = a_{23} = a_{24} \\
(3) & b_{13} = 2(b_{11} + b_{12}),\ b_{14} \neq b_{13} \\
(4) & b_{21} = b_{22} = b_{23} = b_{24} \\
(5) & c_{13} \neq 2(c_{11} + c_{12}) \\
(6) & c_{21} = c_{22} = c_{23} = c_{24}
\end{array} \tag{17}$$

With the A_is defined in this way, no individual two-flat provides estimates of any of the main effects. Thus a design of this type could be resolution IV only if the treatment combinations in the union $T = \bigcup_{i=1}^{4} \{\underline{t} \mid A_i\underline{t} = \underline{c}_i\}$ work together to separate main effects from each other and from interaction effects. Unfortunately, the rigidity of conditions (2), (4), and (6) in (17) ensures that some main effects and interaction effects will be aliased in identical ways on all four two-flats. Suppose, for example, that the $\underline{a}_i$, $\underline{b}_i$, and $\underline{c}_i$, $i = 1, 2, 3, 4$, are chosen according to (17) to give systems of equations (18).

$$(1)\ \begin{bmatrix}1&0&0&1\\0&1&1&1\end{bmatrix}\underline{t}=\begin{bmatrix}0\\1\end{bmatrix}\qquad (2)\ \begin{bmatrix}1&0&1&1\\0&1&1&1\end{bmatrix}\underline{t}=\begin{bmatrix}0\\1\end{bmatrix}$$

$$(3)\ \begin{bmatrix}1&0&2&1\\0&1&1&1\end{bmatrix}\underline{t}=\begin{bmatrix}1\\1\end{bmatrix}\qquad (4)\ \begin{bmatrix}1&0&2&2\\0&1&1&1\end{bmatrix}\underline{t}=\begin{bmatrix}2\\1\end{bmatrix} \tag{18}$$

It follows, for instance, that on every flat the main effects of Factor 2, say F_2, are aliased with the F_3F_4 interaction of Factors 3 and 4 according to the correspondence of levels given in (19).

$$\begin{array}{ccc} F_2(t_2) & & F_3F_4(t_3+t_4) \\ 0 & \leftrightarrow & 1 \\ 1 & \leftrightarrow & 0 \\ 2 & \leftrightarrow & 2 \end{array} \tag{19}$$

That is, for every $\underline{t} = (t_1, t_2, t_3, t_4)'$ in T the following are satisfied:

Whenever $t_2 = 0$, $t_3 + t_4 = 1$.

Whenever $t_2 = 1$, $t_3 + t_4 = 0$.

Whenever $t_2 = 2$, $t_3 + t_4 = 2$.

Thus the design determined by systems of equations (18) does not separate the effects of F_2 from the interaction of F_3 and F_4.

In general, since the preceding example illustrates a consistent occurrence, designs specified by (17) cannot be resolution IV.

The second construction that yields Figure 6 can be achieved in a number of ways, a typical one being specified by the conditions (20).

(1) $a_{11} \neq a_{12}$, $a_{13} = a_{14}$, $a_{11} \neq a_{13}$, $a_{12} \neq a_{13}$

(2) $a_{21} \neq a_{22}$, $a_{23} = a_{24}$, $a_{21} \neq a_{23}$, $a_{22} \neq a_{23}$

(3) $b_{22} = 2b_{11} + b_{21} + b_{12}$, $b_{13} = 2b_{11} + 2b_{12}$, $b_{23} = b_{11} + b_{21} + 2b_{12}$, $b_{14} \neq b_{13}$, $b_{24} = 2b_{11} + b_{21} + b_{14}$ (20)

(4) $c_{22} = 2c_{11} + c_{21} + c_{12}$, $c_{13} \neq 2c_{11} + 2c_{12}$, $c_{23} = 2c_{11} + c_{21} + c_{13}$, $c_{24} = 2c_{11} + c_{21} + c_{14}$

Designs of the type (20) resemble those of type (17) in that the individual flats do not guarantee estimability of any of the main effects. Moreover, in any specific instance it is easy to see that the flats do not work together to create a resolution IV design. For example, if the $\underline{a}_i$, $\underline{b}_i$, and $\underline{c}_i$, i = 1, 2, 3, 4, are chosen according to (20) to give systems of equations

$$
\begin{aligned}
&(1) \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \underline{t} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad (2) \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \underline{t} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\
&(3) \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} \underline{t} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad (4) \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 2 \end{bmatrix} \underline{t} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\end{aligned} \tag{21}
$$

it follows that main effects F_1 are aliased with interaction effects F_3F_4 by the same correspondence on all four planes. Thus Design (21), like Design (18) cannot be resolution IV.

A second design constructed according to (20) consists of solutions to systems of equations (22).

$$
\begin{aligned}
&(1) \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \underline{t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (2) \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \underline{t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&(3) \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \underline{t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (4) \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \underline{t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned} \tag{22}
$$

The aliasing patterns determined by these four flats are not as easy to analyze as those of (21). However, to cite one example, on the first flat the main effects for F_1 are aliased with interaction $F_1F_4^2$ according to

$F_1(t_1)$		$F_3F_4^2(t_1 + 2t_2)$
0	$\leftrightarrow$	0
1	$\leftrightarrow$	2
2	$\leftrightarrow$	1

while on the second flat the main effects of F_2 are aliased with $F_1F_4^2$ by the same correspondence. This occurrence, along with other similar correspondences, suggested that Design (22) is not resolution IV, a result that has been verified by direct computer check.

In summary, although the discussion of the preceding paragraphs does not constitute a proof that no 21 run 3^4 resolution IV design can be constructed from four intersecting two-flats, it does indicate that this is probably the case. Moreover, it surely points out the need for a means of analyzing how the choice of A_i and $\underline{c}_i$, $i = 1, \ldots, n$, affect estimability in a design of the form $T = \bigcup_{i=1}^{n} \{\underline{t} \mid A_i\underline{t} = \underline{c}_i\}$.

4. Estimability of Effects

The examples and discussion of the previous section illustrate clearly how the estimability of effects from design $T = \bigcup_{i=1}^{n} \{\underline{t} | A_i \underline{t} = \underline{c}_i\}$ depends not only on the individual A_is, but also on the structure of intersection resulting from choice of $\underline{c}_i$, $i = 1, 2, \ldots, n$. If the A_is are as specified by Theorem 2 of Section 2, it is, of course, possible to estimate all main effects using just one flat at a time. However, the estimability of μ that occurs if $\underline{c}_1 = \underline{c}_2 = \ldots = \underline{c}_n = \underline{0}$ is provided not by any individual flat, but rather by the joint application of all treatment combinations in a union of n flats. Similarly, the foldover type designs summarized in Example 8 of Section 3 are all resolution IV although estimability of certain main effects is not guaranteed by considering the flats one at a time.

The present section introduces an approach to the analysis of the aliasing structure for designs formed as the union of a number of flats. The discussion includes examples which illustrate use of the basic procedures.

For the 3^n experiment consider again the general design

$$T = \bigcup_{i=1}^{r} \{\underline{t} \mid A_i \underline{t} = \underline{c}_i\}, \tag{23}$$

where each A_i is an $m_i \times n$ matrix of rank m_i. For each i, $i = 1, \ldots, r$, the model equations that correspond to the $N_i = s^{n-m_i}$ treatment combinations specified by $A_i \underline{t} = \underline{c}_i$ are

$$E(\underline{Y}_i) = X_i \underline{\beta} \tag{24}$$

where X_i is an $N_i \times \nu$ matrix of known constants, and $\underline{\beta}$ is a $\nu \times 1$ vector of non-negligible unknown parameters. Thus the model for the N observations corresponding to treatment combinations in (23) can be partitioned as

$$E[\underline{Y}] = E\begin{bmatrix} \underline{Y}_1 \\ \underline{Y}_2 \\ \vdots \\ \underline{Y}_r \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \end{bmatrix} \underline{\beta} = X\underline{\beta}. \tag{25}$$

For every i, i = 1, ..., r, the coefficient matrix A_i partitions the parameters in $\underline{\beta}$ into a number of alias sets, say $S_{i1}, \ldots, S_{iu_i}$. For example, in a 3^4 experiment involving factors F_1, F_2, F_3, and F_4 the coefficient matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

produces the five alias sets $S_1 = \{\mu\}$, $S_2 = \{F_1, F_2F_3, F_3F_4, F_2F_4^2\}$, $S_3 = \{F_2, F_1F_3^2, F_1F_4^2, F_3F_4^2\}$, $S_4 = \{F_3, F_1F_4, F_2F_4, F_1F_2^2\}$, and $S_5 = \{F_4, F_1F_2, F_1F_3, F_2F_3^2\}$, where μ has one degree of freedom and every other effect has two degrees of freedom corresponding to the linear and quadratic contrasts that were discussed in Section 1.

Suppose now that E is an effect of interest and that estimability of the linear (E_L) and quadratic (E_Q) components of E is not guaranteed by any one flat in (3.23). Let $S_i^E = \{E = E_{i1}, E_{i2}, \ldots, E_{i\ell_i}\}$ be the alias set including E that is generated by coefficient matrix A_i, i = 1, ..., r, and assume that each E_{ij}, j = 1, ..., ℓ_i, i = 1, ..., r, carries two degrees of freedom. Then if E_{ij} and $E_{ij'}$ are any two

members of S_i^E, it is possible to transform the levels of E_{ij} to the levels of $E_{ij'}$ by applying one of the permutations in (26).

$$\begin{array}{rccc@{\qquad}rccc}
 & E_{ij} & & E_{ij'} & & E_{ij} & & E_{ij'} \\
 & 0 & \rightarrow & 0 & & 0 & \rightarrow & 1 \\
e: & 1 & \rightarrow & 1 & (01): & 1 & \rightarrow & 0 \\
 & 2 & \rightarrow & 2 & & 2 & \rightarrow & 2 \\
 & 0 & \rightarrow & 1 & & 0 & \rightarrow & 2 \\
(012): & 1 & \rightarrow & 2 & (02): & 1 & \rightarrow & 1 \\
 & 2 & \rightarrow & 0 & & 2 & \rightarrow & 0 \\
 & 0 & \rightarrow & 2 & & 0 & \rightarrow & 0 \\
(021): & 1 & \rightarrow & 0 & (12): & 1 & \rightarrow & 2 \\
 & 2 & \rightarrow & 1 & & 2 & \rightarrow & 1
\end{array} \tag{26}$$

Moreover, if X_i^E represents the submatrix of X_i consisting of the $2\ell_i$ columns that correspond to linear and quadratic components for E_{ij}, $j = 1, \ldots, \ell_i$, it is clear that X_i^E is of rank two since exactly two degrees of freedom are associated with each effect in S_i^E. Suppose specifically that linear and quadratic contrasts are specified by the orthogonal polynomial coefficients

Effect / Level	Linear	Quadratic
0	-1	1
1	0	-2
2	1	1

.

Then if E_{ij} and $E_{ij'}$ in S_i^E are related by permutation g of (26), and if $X_{i;jj'}^E$ consists of the four columns of X_i^E associated with E_{ij} and $E_{ij'}$, one finds that

$$X_{i;jj'}^E \; Pg = 0,$$

where Pg is determined by permutation g in accordance with (27).

$\underline{g}$	$\underline{Pg}$	$\underline{g}$	$\underline{Pg}$	
e	$\begin{matrix} 2 & 0 \\ 0 & 2 \\ -2 & 0 \\ 0 & -2 \end{matrix}$	(01)	$\begin{matrix} 2 & 0 \\ 0 & 2 \\ -1 & -3 \\ -1 & 1 \end{matrix}$	
(012)	$\begin{matrix} 2 & 0 \\ 0 & 2 \\ 1 & 3 \\ -1 & 1 \end{matrix}$	(02)	$\begin{matrix} 2 & 0 \\ 0 & 2 \\ 2 & 0 \\ 0 & -2 \end{matrix}$	(27)
(021)	$\begin{matrix} 2 & 0 \\ 0 & 2 \\ 1 & -3 \\ 1 & 1 \end{matrix}$	(12)	$\begin{matrix} 2 & 0 \\ 0 & 2 \\ -1 & 3 \\ 1 & 1 \end{matrix}$	

It is instructive to consider first the special case for which $A_i = B$, a constant m x n coefficient matrix independent of i. Then design (23) becomes

$$T = \bigcup_{i=1}^{r} \{\underline{t} \mid B\underline{t} = \underline{d}_i\}, \qquad (28)$$

each flat consists of $N_i = 3^{n-m}$ treatment combinations, and the resulting union of parallel (n - m)-flats includes a total $N = (r)(3^{n-m})$ treatment combinations. Moreover, $S_i^E = S^E = \{E = E_1, E_2, \ldots, E_\ell\}$ is the same for all r flats.

Consider for each i, i = 1, ..., r, the null space of X_i^E. That is, given that S^E includes ℓ effects, each with two degrees of freedom, consider the space of all $2\ell \times 1$ vectors $\underline{w}$ such that $X_i^E\underline{w} = \underline{0}$. Theorems 3 and 4, which follow, establish a means for investigating the estimability of E_L and E_Q through consideration of the null spaces for X_i^E, i = 1, ..., r.

Theorem 3. Let W_i^E be a $2\ell \times 2(\ell - 1)$ matrix of full column rank such that $X_i^E W_i^E = 0$, where 0 represents an $N_i \times 2(\ell - 1)$ matrix of all zeros. Then the column space of W_i^E, denoted $C(W_i^E)$, is the same as the null space of X_i^E.

Proof. Since X_i^E is of rank two, if $\underline{w}$ is a solution to $X_i^E\underline{w} = \underline{0}$, it is possible to solve for two components of $\underline{w}$ in terms of $2\ell - 2$ arbitrary components. Thus the null space of X_i^E has dimension $2\ell - 2$. If W_i^E is of full column rank and such that $X_i^E W_i^E = 0$, the columns of W_i^E clearly form a basis for the null space of X_i^E. The proof is complete.

Theorem 4. The linear and quadratic components for E are not estimable from the runs of design (28) if and only if there exists a vector $\underline{w} \in \bigcap_{i=1}^{r} C(W_i^E)$ such that the components of $\underline{w}$ which correspond to E_L and E_Q are nonzero.

Proof. In terms of model (25) any component of $\underline{\beta}$ is estimable if and only if the corresponding column of X is linearly independent of all remaining columns of X. Since the structure of design (28) guarantees that E_L and E_Q can be aliased only with effects in S^E, it suffices to consider only the submatrix of X consisting of columns for the linear and quadratic components of effects in S^E. Partition this submatrix as

$$X^E = \begin{bmatrix} X_1^E \\ X_2^E \\ \vdots \\ X_r^E \end{bmatrix},$$

and let the first two columns of X^E correspond respectively to E_L and E_Q. Now E_L is not estimable if and only if there exists a $2\ell \times 1$ vector $\underline{w} = [w_1, w_2, \ldots, w_{2\ell}]'$ such that $X^E\underline{w} = \underline{0}$ and in which $w_1 \neq 0$. Clearly such a $\underline{w}$ must be in $\cap C(W_i^E)$. A similar argument holds for E_Q, and the proof is complete.

In practice the matrix W_i^E which provides the basis for the null space of X_i^E can be formed in any of a number of ways. Two of the most useful forms are specified by Definitions 2 and 3.

Definition 2. W_i^E, a basis for the null space of X_i^E, is said to be in standard form with respect to effect E_k, where E_k may or may not be equal to E, if W_i^E is written as

$$\left[\begin{array}{c|c|c|c|c|c|c}
\begin{smallmatrix}2&0\\0&2\end{smallmatrix} & \begin{smallmatrix}2&0\\0&2\end{smallmatrix} & \cdots & \begin{smallmatrix}2&0\\0&2\end{smallmatrix} & \begin{smallmatrix}2&0\\0&2\end{smallmatrix} & \cdots & \begin{smallmatrix}2&0\\0&2\end{smallmatrix} \\
\hline
Z_{k1} & 0 & & 0 & 0 & & 0 \\
0 & Z_{k2} & & \cdot & \cdot & & \cdot \\
\cdot & 0 & \ddots & \cdot & \cdot & & \cdot \\
\cdot & \cdot & & 0 & \cdot & & \cdot \\
\cdot & \cdot & & Z_{k,k-1} & 0 & & \cdot \\
\cdot & \cdot & & 0 & Z_{k,k+1} & & \cdot \\
\cdot & \cdot & & \cdot & 0 & \ddots & \cdot \\
\cdot & \cdot & & \cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot & \cdot & & 0 \\
0 & 0 & & 0 & 0 & & Z_{k\ell}
\end{array}\right]$$

where each 0 block is a 2 x 2 matrix of zeros and each Z_{kj} is determined from (27) according to the permutation required to change the levels of E_k to the levels of E_j, $j = 1, \ldots, \ell$, $j \neq k$.

Definition 3. W_i^E, a basis for the null space of X_i^E, is said to be in diagonal form if W_i^E is written as

$$
\left[\begin{array}{c|c|c|c|c}
\begin{matrix}2&0\\0&2\end{matrix} & 0 & 0 & \cdots & 0 \\
\hline
Z_{12} & \begin{matrix}2&0\\0&2\end{matrix} & 0 & & \cdot \\
0 & Z_{23} & \begin{matrix}2&0\\0&2\end{matrix} & & \cdot \\
\cdot & 0 & Z_{34} & & \cdot \\
\cdot & \cdot & 0 & & 0 \\
\cdot & \cdot & \cdot & & \begin{matrix}2&0\\0&2\end{matrix} \\
\hline
0 & 0 & 0 & & Z_{\ell-1,\ell}
\end{array}\right]
$$

where for some ordering of the effects in S^E, $Z_{k,k+1}$ is determined from (27) according to the permutation required to change the levels of E_k to the levels of E_{k+1}, $k = 1, \ldots, \ell - 1$.

<u>Example 9.</u> For the 3^3 experiment involving factors F_1, F_2, and F_3 consider design $T = \bigcup_{i=1}^{7} \{\underline{t} \mid B\underline{t} = \underline{d}_i\}$, where

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

and $\underline{d}_1 = (0, 0)'$, $\underline{d}_2 = (0, 1)'$, $\underline{d}_3 = (0, 2)'$, $\underline{d}_4 = (1, 0)'$, $\underline{d}_5 = (2,0)'$, $\underline{d}_6 = (1, 1)'$, and $\underline{d}_7 = (2, 2)'$. The resulting design T consists of the 21 treatment combinations shown in (29), where spacing indicates the seven individual flats.

$$T = \begin{bmatrix} 012 & 012 & 012 & 012 & 012 & 012 & 012 \\ 012 & 012 & 012 & 120 & 201 & 120 & 201 \\ 012 & 120 & 201 & 012 & 012 & 120 & 201 \end{bmatrix} \tag{29}$$

Coefficient matrix B produces alias set $S^{F_1} = S^{F_2} = S^{F_3} = S = \{F_1, F_2, F_3, F_1F_2, F_1F_3, F_2F_3\}$ and the $W_i^{F_1} = W_i^{F_2} = W_i^{F_3} = W_i$, $i = 1, \ldots, 7$, each written in diagonal form, are as given in (30). Unspecified entries in each W_i are all equal to 0.

$$\begin{bmatrix}
2 & 0 & & & & & & & & \\
0 & 2 & & & & & & & & \\
-2 & 0 & 2 & 0 & & & & & & \\
0 & -2 & 0 & 2 & & & & & & \\
 & & -2 & 0 & 2 & 0 & & & & \\
 & & 0 & -2 & 0 & 2 & & & & \\
 & & & & -1 & 3 & 2 & 0 & & \\
 & & & & 1 & 1 & 0 & 2 & & \\
 & & & & & & -2 & 0 & 2 & 0 \\
 & & & & & & 0 & -2 & 0 & 2 \\
 & & & & & & & & -2 & 0 \\
 & & & & & & & & 0 & -2
\end{bmatrix}
\qquad
\begin{bmatrix}
2 & 0 & & & & & & & & \\
0 & 2 & & & & & & & & \\
-2 & 0 & 2 & 0 & & & & & & \\
0 & -2 & 0 & 2 & & & & & & \\
 & & 1 & 3 & 2 & 0 & & & & \\
 & & -1 & 1 & 0 & 2 & & & & \\
 & & & & -1 & -3 & 2 & 0 & & \\
 & & & & -1 & 1 & 0 & 2 & & \\
 & & & & & & 1 & 3 & 2 & 0 \\
 & & & & & & -1 & 1 & 0 & 2 \\
 & & & & & & & & -2 & 0 \\
 & & & & & & & & 0 & -2
\end{bmatrix}$$

$$W_1 \qquad\qquad\qquad\qquad W_2$$

$$\begin{bmatrix}
2 & 0 & & & & & & & & \\
0 & 2 & & & & & & & & \\
-2 & 0 & 2 & 0 & & & & & & \\
0 & -2 & 0 & 2 & & & & & & \\
 & & 1 & -3 & 2 & 0 & & & & \\
 & & 1 & 1 & 0 & 2 & & & & \\
 & & & & 2 & 0 & 2 & 0 & & \\
 & & & & 0 & -2 & 0 & 2 & & \\
 & & & & & & 1 & -3 & 2 & 0 \\
 & & & & & & 1 & 1 & 0 & 2 \\
 & & & & & & & & -2 & 0 \\
 & & & & & & & & 0 & -2
\end{bmatrix}
\qquad
\begin{bmatrix}
2 & 0 & & & & & & & & \\
0 & 2 & & & & & & & & \\
1 & 3 & 2 & 0 & & & & & & \\
-1 & 1 & 0 & 2 & & & & & & \\
 & & 1 & -3 & 2 & 0 & & & & \\
 & & 1 & 1 & 0 & 2 & & & & \\
 & & & & -1 & -3 & 2 & 0 & & \\
 & & & & 1 & 1 & 0 & 2 & & \\
 & & & & & & 1 & -3 & 2 & 0 \\
 & & & & & & 1 & 1 & 0 & 2 \\
 & & & & & & & & 1 & 3 \\
 & & & & & & & & -1 & 1
\end{bmatrix}$$

$$W_3 \qquad\qquad\qquad\qquad W_4$$

$$\begin{bmatrix}
2 & 0 & & & & & & & & \\
0 & 2 & & & & & & & & \\
1 & -3 & 2 & 0 & & & & & & \\
1 & 1 & 0 & 2 & & & & & & \\
 & & 1 & 3 & 2 & 0 & & & & \\
 & & -1 & 1 & 0 & 2 & & & & \\
 & & & & 2 & 0 & 2 & 0 & & \\
 & & & & 0 & -2 & 0 & -2 & & \\
 & & & & & & 1 & 3 & 2 & 0 \\
 & & & & & & -1 & 1 & 0 & 2 \\
 & & & & & & & & 1 & -3 \\
 & & & & & & & & 1 & 1
\end{bmatrix}
\qquad
\begin{bmatrix}
2 & 0 & & & & & & & & \\
0 & 2 & & & & & & & & \\
1 & 3 & 2 & 0 & & & & & & \\
-1 & 1 & 0 & 2 & & & & & & \\
 & & -2 & 0 & 2 & 0 & & & & \\
 & & 0 & -2 & 0 & 2 & & & & \\
 & & & & 2 & 0 & 2 & 0 & & \\
 & & & & 0 & -2 & 0 & 2 & & \\
 & & & & & & -2 & 0 & 2 & 0 \\
 & & & & & & 0 & -2 & 0 & 2 \\
 & & & & & & & & 1 & 3 \\
 & & & & & & & & -1 & 1
\end{bmatrix}
\qquad (30)$$

$$W_5 \qquad\qquad\qquad\qquad W_6$$

$$
\begin{bmatrix}
2 & 0 & & & & & & & & \\
0 & 2 & & & & & & & & \\
1 & -3 & 2 & 0 & & & & & & \\
1 & 1 & 0 & 2 & & & & & & \\
 & & -2 & 0 & 2 & 0 & & & & \\
 & & 0 & -2 & 0 & 2 & & & & \\
 & & & & -1 & -3 & 2 & 0 & & \\
 & & & & -1 & 1 & 0 & 2 & & \\
 & & & & & & -2 & 0 & 2 & 0 \\
 & & & & & & 0 & -2 & 0 & 2 \\
 & & & & & & & & 1 & -3 \\
 & & & & & & & & 1 & 1
\end{bmatrix}
$$

$$W_7$$

By Theorem 4 for k = 1, 2, or 3 the linear and quadratic effects of factor F_k are not estimable if and only if there exists a vector $\underline{w} = [w_1, w_2, \ldots, w_{12}]'$ with nonzero components corresponding to $F_{k,L}$ and $F_{k,Q}$ such that $\underline{w}$ can be written as some linear combination of the columns of W_i, say $W_i\underline{\alpha}_i$, for each i, i = 1, ..., 7. Thus to prove the estimability of $F_{k,L}$ and $F_{k,Q}$ it suffices to show that

$$\underline{w} = W_1\underline{\alpha}_1 = W_2\underline{\alpha}_2 = \ldots = W_7\underline{\alpha}_7$$

forces the components of $\underline{w}$ which correspond to $F_{k,L}$ and $F_{k,Q}$ to be zero. Now if $\underline{\alpha}_1 = [\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1,10}]'$, $\underline{w} = W_1\underline{\alpha}_1 = W_6\underline{\alpha}_6 = W_7\underline{\alpha}_7$ implies that

$$
\underline{\alpha}_6 = \left(\tfrac{1}{2}\right)
\begin{bmatrix}
2\alpha_{11} \\
2\alpha_{12} \\
-3\alpha_{11} - 3\alpha_{12} + 2\alpha_{13} \\
\alpha_{11} - 3\alpha_{12} + 2\alpha_{14} \\
-3\alpha_{11} - 3\alpha_{12} + 2\alpha_{15} \\
\alpha_{11} - 3\alpha_{12} + 2\alpha_{16} \\
3\alpha_{11} + 3\alpha_{12} - 3\alpha_{15} + 3\alpha_{16} + 2\alpha_{17} \\
\alpha_{11} - 3\alpha_{12} + \alpha_{15} + 3\alpha_{16} + 2\alpha_{18} \\
3\alpha_{11} + 3\alpha_{12} - 3\alpha_{15} + 3\alpha_{16} + 2\alpha_{19} \\
\alpha_{11} - 3\alpha_{12} + \alpha_{15} + 3\alpha_{16} - 2\alpha_{1,10}
\end{bmatrix}
\quad \text{and}
$$

$$\underline{\alpha}_7 = \left(\frac{1}{2}\right) \begin{bmatrix} 2\alpha_{11} \\ 2\alpha_{12} \\ -3\alpha_{11} + 3\alpha_{12} + 2\alpha_{13} \\ -\alpha_{11} - 3\alpha_{12} + 2\alpha_{14} \\ -3\alpha_{11} + 3\alpha_{12} + 2\alpha_{15} \\ -\alpha_{11} - 3\alpha_{12} + 2\alpha_{16} \\ -3\alpha_{11} - 3\alpha_{12} + 6\alpha_{16} + 2\alpha_{17} \\ -\alpha_{11} + 3\alpha_{12} + 2\alpha_{15} + 2\alpha_{18} \\ -3\alpha_{11} - 3\alpha_{12} + 6\alpha_{16} + 2\alpha_{19} \\ -\alpha_{11} + 3\alpha_{12} + 2\alpha_{15} + 2\alpha_{1,10} \end{bmatrix} .$$

Thus $w_{11} = -2\alpha_{19} = 3\alpha_{11} - 3\alpha_{12} + 6\alpha_{16} + \alpha_{19} + 3\alpha_{1,10} = -6\alpha_{12} - 3\alpha_{15} + \alpha_{19} - 3\alpha_{1,10}$ and $w_{12} = -2\alpha_{1,10} = -\alpha_{11} - 3\alpha_{12} + 2\alpha_{15} - \alpha_{19} + \alpha_{1,10} = -2\alpha_{11} + \alpha_{15} + 3\alpha_{16} + \alpha_{19} + 2\alpha_{1,10}$, so the equations of (31) must be satisfied.

$$\begin{bmatrix} 3 & 3 & 3 & 3 & 0 & 6 \\ 3 & -3 & 0 & 6 & 3 & 3 \\ 1 & -3 & 1 & -3 & -2 & 0 \\ -1 & -3 & 2 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{15} \\ \alpha_{16} \\ \alpha_{19} \\ \alpha_{1,10} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{31}$$

Elementary operations can be used to reduce (31) to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & -2 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{15} \\ \alpha_{16} \\ \alpha_{19} \\ \alpha_{1,10} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which implies that $\alpha_{11} = \alpha_{12} = 0$. Thus the linear and quadratic effects for F_1 are estimable. Similarly, $\underline{w} = W_1\underline{\alpha}_1 = W_2\underline{\alpha}_2 = W_3\underline{\alpha}_3$ implies that $F_{2,L}$ and $F_{2,Q}$ are estimable, and $\underline{w} = W_1\underline{\alpha}_1 = W_4\underline{\alpha}_4 = W_5\underline{\alpha}_5$ implies that $F_{3,L}$ and $F_{3,Q}$ are estimable.

Two additional results concerning the aliasing structure for constructions of type (28) are given in Theorems 5 and 6.

Theorem 5. Let the 3^n design $T = \bigcup_{i=1}^{r} \{\underline{t} \mid B\underline{t} = \underline{d}_i\}$, where B is an $(n-1) \times n$ matrix of the form

$$\begin{bmatrix} 2 & \\ 2 & \\ 2 & I_{n-1} \\ \vdots & \\ 2 & \end{bmatrix} \tag{32}$$

and $\underline{d}_i = (d_{i1}, d_{i2}, \ldots, d_{i,n-1})'$, $i = 1, \ldots, r$. For any k, $k = 2, \ldots, n$, the linear and quadratic effects of the kth factor are estimable from the runs of T if the set of all $\underline{d}_i$, $i = 1, \ldots, r$, includes a subset of size three, say $\underline{d}_s$, $\underline{d}_u$ and $\underline{d}_v$, in which $d_{s\gamma} = d_{u\gamma} = d_{v\gamma}$ for $\gamma = 1, \ldots, k-1, k+1, \ldots, n-1$, while the kth component ranges over the three values in GF(3).

Proof. Consider the set of treatment combinations $T_k = \bigcup_{i=s,u,v} \{\underline{t} \mid B\underline{t} = \underline{d}_i\}$, where B is defined by (23). Since $\underline{d}_s$, $\underline{d}_u$, and $\underline{d}_v$ are equal except for the kth component, which ranges over all three possible values, the levels of Factor k are not associated with the levels of any other effect by one of the permutations in (26). Thus given the model $E(\underline{Y}_k) = X_k\underline{\beta}$, where $\underline{Y}_k$ is the vector of random observations corresponding to treatment combinations in T_k, it is not possible to involve the columns associated with linear and quadratic F_k in a linear combination equal to $\underline{0}$. Thus in the model for the full design T the columns of X associated with F_k are linearly independent of all remaining columns of X, so the linear and quadratic effects of F_k are estimable. The proof is complete.

Suppose that the conditions of Theorem 5 are satisfied for each k, k = 2, ..., n. If the condition is satisfied for k = 1 after taking the appropriate linear combination of the rows of B and the elements of $\underline{d}_i$, i=1,..., r then the design is resolution IV. Example 10 illustrates how this approach can be used to consider estimability in a 3^4 foldover design of the type discussed previously in Section 3.

Example 10. A 3^4 foldover design for a factorial design involving factors F_1, F_2, F_3, and F_4 was given in Section 3 as a union of four intersecting flats. This design can be expressed in the format of Theorem 5 as $T = \bigcup_{i=1}^{9} \{\underline{t} \mid B\underline{t} = \underline{d}_i\}$, where

$$B = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

and the $\underline{d}_i$, i = 1, ..., 9, are the columns of the matrix (33).

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} = [\underline{d}_1, \underline{d}_2, \ldots, \underline{d}_9] \tag{33}$$

Estimability of the main effects of F_2, F_3, and F_4 is guaranteed by Theorem 5 as follows: F_2 by $\underline{d}_1$, $\underline{d}_6$ and $\underline{d}_7$; F_3 by $\underline{d}_1$, $\underline{d}_4$, and $\underline{d}_5$; F_3 by $\underline{d}_1$, $\underline{d}_2$, and $\underline{d}_3$. Since elementary row transformations will transform

$$[B \mid \underline{d}_1\ \underline{d}_8\ \underline{d}_9] = \left[\begin{array}{cccc|ccc} 2 & 1 & 0 & 0 & 0 & 1 & 2 \\ 2 & 0 & 1 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 & 0 & 1 & 2 \end{array}\right]$$

to $$\begin{bmatrix} 1 & 2 & 0 & 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

the main effects of F_1 are estimable also. Thus the design is resolution IV.

Theorem 6. Suppose that $\underline{d}_1, \ldots, \underline{d}_r$ of (28) are selected so that for some E_k in $S^E = \{E_1, E_2, \ldots, E_\ell\}$ the following conditions are satisfied.

1. For each $j \neq k$, the permutation of (26) that relates E_k to E_j is the same for each of $\underline{d}_1, \underline{d}_2, \ldots, \underline{d}_r$.
2. The list of r permutations relating E_k to E for $\underline{d}_1, \ldots, \underline{d}_r$, respectively, includes at least two distinct permutations.

Then design (28) ensures the estimability of the linear and quadratic effects of E.

Proof. If necessary, reorder the elements of S^E so that $E_k = E_1$ and $E = E_2$. For each i, i = 1, ..., r, write a basis W_i for the null space of X_i^E in standard form with respect to $E_k = E_1$. That is,

$$W_i = \left[\begin{array}{cc|cc|c|cc} 2 & 0 & 2 & 0 & & 2 & 0 \\ 0 & 2 & 0 & 2 & \cdots & 0 & 2 \\ \hline \multicolumn{2}{c|}{V_{i2}} & \multicolumn{2}{c|}{0} & & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{V_{i3}} & & \multicolumn{2}{c}{\cdot} \\ \multicolumn{2}{c|}{\cdot} & \multicolumn{2}{c}{0} & \ddots & \multicolumn{2}{c}{\cdot} \\ \multicolumn{2}{c|}{\cdot} & \multicolumn{2}{c}{\cdot} & & \multicolumn{2}{c}{\cdot} \\ \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{0} & & \multicolumn{2}{c}{V_{i\ell}} \end{array}\right] = \left[\begin{array}{cc|cc|c|cc} 2 & 0 & 2 & 0 & & 2 & 0 \\ 0 & 2 & 0 & 2 & \cdots & 0 & 2 \\ \hline \multicolumn{2}{c|}{V_{i2}} & \multicolumn{2}{c|}{0} & \cdots & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{0} & & & & & \\ \multicolumn{2}{c|}{\cdot} & & & V_i & & \\ \multicolumn{2}{c|}{\cdot} & & & & & \\ \multicolumn{2}{c|}{0} & & & & & \end{array}\right]$$

where V_{ij}, j = 2, 3, ..., ℓ is determined by (27) according to the permutation relationship between $E_k = E_1$ and E_j. Now $\underline{w} \in \bigcap_{i=1}^{r} C(W_i)$

implies there exist $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_r$ such that

$$\underline{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ \vdots \\ w_{2\ell} \end{bmatrix} = W_i \underline{a}_i = \left[\begin{array}{c|ccc} 2I & 2I & \ldots & 2I \\ V_{i2} & 0 & \ldots & 0 \\ \hline 0 & & V_i & \end{array}\right] \left[\begin{array}{c} a_{i1} \\ a_{i2} \\ \hline \underline{a}_{iv} \\ \end{array}\right] \quad i = 1, \ldots, r.$$

Since $V_i \underline{a}_{iv} = [w_5, \ldots, w_{2\ell}]'$ for all i, $\underline{a}_{iv} = V_i^{-1}[\ _5, \ldots, \ _{2\ell}]'$ for all i, i = 1, ..., r. Thus

$$\begin{bmatrix} a_{i1} \\ a_{i2} \end{bmatrix} = -2[I \ldots I]\, \underline{a}_{iv}$$
$$= -2[I \ldots I]V^{-1} \begin{bmatrix} w_5 \\ \vdots \\ w_{2\ell} \end{bmatrix}$$

for all i. Now

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = V_{i2} \begin{bmatrix} a_{i1} \\ a_{i2} \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} a_{i1} \\ a_{i2} \end{bmatrix} = V_{i2}^{-1} \begin{bmatrix} w_3 \\ w_4 \end{bmatrix}, \ i = 1, \ldots, r.$$

Thus for i ≠ i'

$$\begin{bmatrix} a_{i1} \\ a_{i2} \end{bmatrix} = \begin{bmatrix} a_{i'1} \\ a_{i'2} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{i1} \\ a_{i2} \end{bmatrix} = 0, \ i = 1, \ldots, r$$
$$\Rightarrow V_{i2}^{-1} \begin{bmatrix} a_{i1} \\ a_{i2} \end{bmatrix} = 0, \ i = 1, \ldots, r,$$

so $\underline{w} \in \cap\, C(W_i)$ implies $w_3 = w_4 = 0$, and it follows that the main effects of E are estimable. The proof is complete.

Example 11. For the 3^4 experiment involving factors F_1, F_2, F_3, and F_4 consider $T = \bigcup_{i=1}^{3} \{\underline{t} \mid B\underline{t} = \underline{d}_i\}$, where

$$B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

and $\underline{d}_1 = (0, 0)'$, $\underline{d}_2 = (1, 0)'$, $\underline{d}_3 = (2, 0)'$. Coefficient matrix B generates the alias set $S^{F_1} = \{F_1, F_2F_3, F_3F_4, F_2F_4^2\}$, and

$$T = \begin{bmatrix} 021112200 & 102220011 & 210001122 \\ 012012012 & 012012012 & 012012012 \\ 012100221 & 012100221 & 012100221 \\ 000121212 & 000121212 & 000121212 \end{bmatrix} \tag{34}$$

where the spacing indicates the treatment combinations on the individual flats. From (34), one sees that the permutation relating F_2F_3 to F_1 is e on flat 1, (012) on flat 2, and (021) on flat 3. Since F_2F_3 is related to F_3F_4 and to $F_2F_4^2$ by the (12) permutation for all runs in T, Theorem 6 guarantees that T provides estimates for the linear and quadratic effects of F_1.

The preceding discussion of the aliasing structure for designs formed as the union of r parallel flats extends readily to designs specified by (23) as a union of intersecting or nonintersecting flats. In this more general case, however, the alias set $S_i^E = \{E_{i1} = E, E_{i2}, \ldots, E_{i\ell_i}\}$ that includes effect E depends on A_i, and X_i^E is taken to be the submatrix of X_i consisting of all columns that correspond to effects in

$$S^Q = \bigcup_{i=1}^{r} \bigcup_{j=1}^{\ell_i} S_i^{E_{ij}},$$

where $Q = \bigcup_{i=1}^{r} S_i^E$. That is, S^Q consists of all effects that are aliased with an effect in Q on any flat, i = 1, ..., r. It is, of course, permissible to eliminate from S^Q any effect that is known to be estimable from the treatment combinations on one of the individual flats. For each i, i = 1, ..., r, the W_i^E matrix that supplies a basis for the null space of X_i^E is composed of one or more matrices of the type specified in Definitions 2 and 3, depending on how many of the alias sets generated by A_i are included in S^Q. The notions of this paragraph are illustrated in Example 12.

Example 12. Design 3 of Example 8, Section 3, is

$T = \bigcup_{i=1}^{4} \{\underline{t} \mid A_i\underline{t} = \underline{0}\}$, where

$$A_1 = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix},$$

all over GF(3). Theorem 1 guarantees the estimability of the main effects of Factors F_1, F_2, and F_3 individually from flats 1, 2, and 3, respectively. Estimability of the linear and quadratic components of F_4, on the other hand, does not follow immediately from consideration of any one flat. In fact, $S_1^{F_4} = \{F_4, F_2, F_3, F_2F_3, F_2F_4, F_3F_4\}$, $S_2^{F_4} = \{F_4, F_1, F_3, F_1F_3, F_1F_4, F_3F_4\}$, $S_3^{F_4} = \{F_4, F_1, F_2, F_1F_2, F_1F_4, F_2F_4\}$, and $S_4^{F_4} = \{F_4, F_1F_3, F_2F_3\}$. Thus $Q = \bigcup_{i=1}^{4} S_i^{F_4} = \{F_1, F_2, F_3, F_4, F_1F_2, F_1F_3, F_1F_4, F_2F_3, F_2F_4, F_3F_4\}$. Since F_1, F_2, and F_3 are known to be estimable from flats 1, 2, and 3 individually, take

$$S^Q = \{F_4, F_1F_2, F_1F_3, F_1F_4, F_2F_3, F_2F_4, F_3F_4\}.$$

The alias sets needed to construct $W_i^{F_4}$, i = 1, 2, 3, 4 and thereby complete the analysis are $S_1^{F_1F_2} = \{F_1F_2, F_1F_3, F_1F_4\}$, $S_2^{F_1F_2} = \{F_1F_2, F_2F_3, F_2F_4\}$, $S_3^{F_2F_3} = \{F_2F_3, F_1F_3, F_3F_4\}$, $S_4^{F_1F_2} = \{F_1F_2, F_3F_4\}$, and $S_4^{F_1F_4} = \{F_1F_4, F_2F_4\}$. The $W_i^{F_4}$, i = 1, 2, 3, 4, which are displayed in (35), are composites of matrices constructed in the standard form specified by Definition 2. The unspecified entries in each $W_i^{F_4}$ are all equal to 0.

$$\begin{array}{c} F_4 \\ \\ F_3F_4 \\ \\ F_2F_3 \\ \\ F_2F_4 \\ \\ F_1F_2 \\ \\ F_1F_3 \\ \\ F_1F_4 \\ \\ \end{array}
\underset{W_1^{F_4}}{\begin{bmatrix}
-1 & 3 & & & & & & & & \\
1 & 1 & & & & & & & & \\
2 & 0 & 2 & 0 & 2 & 0 & & & & \\
0 & 2 & 0 & 2 & 0 & 2 & & & & \\
 & & -2 & 0 & & & & & & \\
 & & 0 & -2 & & & & & & \\
 & & & & -2 & 0 & & & & \\
 & & & & 0 & -2 & & & & \\
 & & & & & & 2 & 0 & 2 & 0 \\
 & & & & & & 0 & 2 & 0 & 2 \\
 & & & & & & -2 & 0 & & \\
 & & & & & & 0 & -2 & & \\
 & & & & & & & & -2 & 0 \\
 & & & & & & & & 0 & -2
\end{bmatrix}} \qquad (35)$$

$$\begin{array}{c} F_4 \\ \\ F_3F_4 \\ \\ F_2F_3 \\ \\ F_2F_4 \\ \\ F_1F_2 \\ \\ F_1F_3 \\ \\ F_1F_4 \\ \\ \end{array}
\underset{W_2^{F_4}}{\begin{bmatrix}
-1 & 3 & & & & & & & & \\
1 & 1 & & & & & & & & \\
2 & 0 & 2 & 0 & 2 & 0 & & & & \\
0 & 2 & 0 & 2 & 0 & 2 & & & & \\
 & & & & & & -2 & 0 & & \\
 & & & & & & 0 & -2 & & \\
 & & & & & & & & -2 & 0 \\
 & & & & & & & & 0 & -2 \\
 & & & & & & 2 & 0 & 2 & 0 \\
 & & & & & & 0 & 2 & 0 & 2 \\
 & & -2 & 0 & & & & & & \\
 & & 0 & -2 & & & & & & \\
 & & & & -2 & 0 & & & & \\
 & & & & 0 & -2 & & & &
\end{bmatrix}}$$

$$
\begin{array}{c}
F_4 \\ \\ F_3F_4 \\ \\ F_2F_3 \\ \\ F_2F_4 \\ \\ F_1F_2 \\ \\ F_1F_3 \\ \\ F_1F_4 \\ \\
\end{array}
\left[\begin{array}{cccccccccc}
-1 & 3 & & & & & & & & \\
1 & 1 & & & & & & & & \\
 & & & & & & & & -2 & 0 \\
 & & & & & & & & 0 & -2 \\
 & & & & & & 2 & 0 & 2 & 0 \\
 & & & & & & 0 & 2 & 0 & 2 \\
 & & & & -2 & 0 & & & & \\
 & & & & 0 & -2 & & & & \\
2 & 0 & 2 & 0 & 2 & 0 & & & & \\
0 & 2 & 0 & 2 & 0 & 2 & & & & \\
 & & & & & & -2 & 0 & & \\
 & & & & & & 0 & -2 & & \\
 & & -2 & 0 & & & & & & \\
 & & 0 & -2 & & & & & &
\end{array}\right]
$$

$W_3^{F_4}$

$$
\begin{array}{c}
F_4 \\ \\ F_3F_4 \\ \\ F_2F_3 \\ \\ F_2F_4 \\ \\ F_1F_2 \\ \\ F_1F_3 \\ \\ F_1F_4 \\ \\
\end{array}
\left[\begin{array}{cccccccc}
-1 & 3 & & & & & & \\
1 & 1 & & & & & & \\
 & & & & & & -2 & 0 \\
 & & & & & & 0 & -2 \\
2 & 0 & 2 & 0 & & & & \\
0 & 2 & 0 & 2 & & & & \\
 & & & & 2 & 0 & & \\
 & & & & 0 & 2 & & \\
 & & & & & & 2 & 0 \\
 & & & & & & 0 & 2 \\
 & & -2 & 0 & & & & \\
 & & 0 & -2 & & & & \\
 & & & & -2 & 0 & & \\
 & & & & 0 & -2 & &
\end{array}\right]
$$

$W_4^{F_4}$

Now $\underline{w} \in \bigcap_{i=1}^{4} C(W_i^{F_4})$ implies there exist $\underline{\alpha}_1$, $\underline{\alpha}_2$, $\underline{\alpha}_3$, and $\underline{\alpha}_4$ such that

$$\underline{w} = W_1^{F_4}\underline{\alpha}_1 = W_2^{F_4}\underline{\alpha}_2 = W_3^{F_4}\underline{\alpha}_3 = W_4^{F_4}\underline{\alpha}_4. \qquad (36)$$

If $\underline{\alpha}_1 = [\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1,10}]$, (36) implies that

$$\underline{\alpha}_2 = \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{17} \\ \alpha_{18} \\ -\alpha_{15} \\ -\alpha_{16} \\ \alpha_{13} \\ \alpha_{14} \\ \alpha_{15} \\ \alpha_{16} \end{bmatrix}, \quad \underline{\alpha}_3 = \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ -\alpha_{15} \\ -\alpha_{16} \\ \alpha_{15} \\ \alpha_{16} \\ \alpha_{17} \\ \alpha_{18} \\ -\alpha_{11} - \alpha_{13} - \alpha_{15} \\ -\alpha_{12} - \alpha_{14} - \alpha_{16} \end{bmatrix}, \text{ and } \underline{\alpha}_5 = \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{17} \\ \alpha_{18} \\ -\alpha_{15} \\ -\alpha_{16} \\ -\alpha_{11} - \alpha_{13} - \alpha_{15} \\ -\alpha_{12} - \alpha_{14} - \alpha_{16} \end{bmatrix}$$

Since $w_3 = 2(\alpha_{11} + \alpha_{13} + \alpha_{15}) = 2(\alpha_{11} + \alpha_{17} - \alpha_{15})$,

$w_5 = -2\alpha_{13} = 2(\alpha_{17} - \alpha_{11} - \alpha_{13} - \alpha_{15}) = 2(\alpha_{11} + \alpha_{17})$, and

$w_9 = 2(\alpha_{13} + \alpha_{15}) = 2(\alpha_{11}) = 2(-\alpha_{11} - \alpha_{13} - \alpha_{15})$,

the equations of (37) must be satisfied.

$$\begin{bmatrix} 0 & 1 & 2 & -1 \\ -1 & 0 & -1 & 1 \\ -2 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{11} \\ \alpha_{13} \\ \alpha_{15} \\ \alpha_{17} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{37}$$

Elementary row operations will reduce(37) to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{11} \\ \alpha_{13} \\ \alpha_{15} \\ \alpha_{17} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which implies that $\alpha_{11} = 0$. A similar consideration of w_4, w_6, and w_{10} implies that $\alpha_{12} = 0$. Thus $w_1 = -\alpha_{11} + 3\alpha_{12} = 0$ and $w_2 = \alpha_{11}+\alpha_{12} = 0$, so the linear and quadratic effects of F_4 are estimable.

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Regular fractional factorial designs for the s^n, $s = p^\alpha$, factorial may be obtained as solutions sets to linear equations of the form $A\underline{t} = \underline{c}$ over GF(s). It is well known how to determine which factorial effects are estimable, and how to construct the alias sets directly from the matrix A. Consider the k sets of equations over GF(s): $A_i\underline{t} = \underline{c}_i$, i=1,2,...,k where A_i is $m_i \times 1$ of rank m_i, and let $T = \bigcup_{i=1}^{k} \{\underline{t} | A_i\underline{t} = \underline{c}_i\}$. If $A_1 = A_2 = \ldots = A_k$, t is called a parallel flats

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fraction, and if not all A_i are equal, T is called an intersecting flats fraction. No unified theory has yet been developed for determining estimability and alias structures from these types of fractions. The purpose of this paper is to present some preliminary results in the development of such a theory.